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Self-organized critical neural networks

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A mechanism for self-organization of the degree of connectivity in model neural networks is studied. Network connectivity is regulated locally on the basis of an order parameter of the global dynamics, which is estimated from an observable at the single synapse level. This principle is studied in a two-dimensional neural network with randomly wired asymmetric weights. In this class of networks, network connectivity is closely related to a phase transition between ordered and disordered dynamics. A slow topology change is imposed on the network through a local rewiring rule motivated by activity-dependent synaptic development: Neighbor neurons whose activity is correlated, on average develop a new connection while uncorrelated neighbors tend to disconnect. As a result, robust self-organization of the network towards the order disorder transition occurs. Convergence is independent of initial conditions, robust against thermal noise, and does not require fine tuning of parameters.

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Information processing in living organisms is often performed by large networks of interacting cells with an overall stunning degree of complexity. How can such networks be efficiently constructed and how can a robust functioning be ensured? The observed complexity of many nervous systems exceeds by far what can be hard coded in the genome [1]. Therefore, developmental principles play a key role in network construction. Furthermore, as learning is a major function of such networks, self-organization and adaptation processes continue throughout the lifetime of a network.

But how can robustness of large dynamical networks be ensured in the face of continuous developmental and adaptive processes? In general, dynamical stability of large networks of dynamical elements and robustness against perturbations are not obtained for free: Model networks with asymmetric connectivity patterns often exhibit regimes of chaotic dynamics with large parameter ranges where network dynamics is not easily controlled [2]. In networks whose central function is information transfer, these regimes would instantly render them useless. Consider, for example, model neural networks with asymmetric synaptic couplings, where a percolation transition between regimes of ordered and disordered dynamics is known [3]. In the disordered phase, which occurs for densely connected networks, already small perturbations percolate through the networks.[1] In such networks, developmental processes that change connectivity always face the risk of driving the network into the highly connected regime (where chaotic dynamics prevails), as long as no explicit mechanism is given that controls the global degree of connectivity.

We here study this question of dynamical robustness of networks in the presence of developmental processes in the context of a simple toy model, an asymmetric neural network combined with simple topology-changing rules. In particular, we ask how a local rewiring mechanism could control global dynamical properties of a large network and actively contribute to avoiding chaotic regimes. While an obvious possibility is a direct feedback of the global dynamical state to the synapses, e.g., controlling synaptic growth rates, we here consider an even simpler mechanism that relies on local information only and, in principle, could be at work in natural systems. We argue that if an order parameter characterizing a global phase transition is accessible at the single synapse level, it can provide the basis for a regulation of global network connectivity solely on the basis of local mechanisms.

Recent models of self-organization of network structures show that it is possible to locally measure a global order parameter connected to the percolation transition of the network, namely, the average activity of a single node over time [5]. Here we will see that, similarly, the average correlation between the activities of two neurons contains information about the global order parameter as well. The network can then use this approximate order parameter to guide the developmental rule. An interesting question is whether self-organization to a critical dynamical transition could occur in a model neural network on the basis of such a correlation. A possible rule is that new synaptic connections preferentially grow between correlated neurons, as suggested by the early ideas of Hebb [6] and the observation of activity-dependent neural development [7]. In the remainder of this paper let us study this problem in the framework of a specific toy model. We will first define a neural network model with a simple mechanism of synaptic development. Then, with numerical studies we will discuss the interplay of dynamics on the network with dynamics of the network topology. Finally, robustness of self-organizing processes in this model and possible implications for biological systems are discussed.

Let us consider a two-dimensional neural network with random asymmetric weights on the lattice. The neighborhood of each neuron is chosen as its Moore neighborhood with

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[1] This is reminiscent of avalanchelike propagation of activity in the brain, which is observed in some diseases of the central nervous system [4].

eight neighbors.[2] The weights w_{ij} are randomly drawn from a uniform distribution $w_{ij} \in [-1,+1]$ and are nonzero between neighbors, only. Note that weights w_{ij} are asymmetric, i.e., in general, $w_{ij} \neq w_{ji}$. Within the neighborhood of a node, a fraction of its weights w_{ij} may be set to 0. The average number of nonzero weights per node is called the average connectivity K of the network (for definiteness count, e.g., the incoming weights at each node, only). The network consists of N neurons with states $\sigma_i = \pm 1$ which are updated in parallel with a stochastic Little dynamics on the basis of inputs received from the neighbor neurons at the previous time step:

$$\mathrm{Prob}[\sigma_i(t+1)=+1]=g_\beta(f_i(t)),$$

$$\mathrm{Prob}[\sigma_i(t+1)=-1]=1-g_\beta(f_i(t)) \tag{1}$$

with

$$f_i(t)=\sum_{j=1}^{N} w_{ij}\sigma_j(t)+\theta_i \tag{2}$$

and

$$g_\beta(f_i(t))=\frac{1}{1+e^{-2\beta f_i(t)}}, \tag{3}$$

with the inverse temperature β and a threshold θ_i. The threshold is chosen here as $\theta_i = -0.1+\gamma$ and includes a small random noise term γ from a Gaussian of width ϵ. This noise term is motivated by the slow fluctuations observed in biological neural systems [8]. With respect to varying either θ or K, the network exhibits a percolation transition between a phase of ordered dynamics, with short transients and short limit cycle attractors, and a phase of chaotic dynamics where the length of dynamical patterns scales exponentially with system size [3,9].

The second part of the model is a slow change of the topology of the network by local rewiring of synaptic weights: If the activity of two neighbor neurons is on average highly correlated (or anticorrelated), they will obtain a common link. If their activity on average is less correlated, they will lose their common link. To be more specific, let us define the average correlation $C_{ij}(\tau)$ of a pair (i,j) of neurons over a time interval τ,

$$C_{ij}(\tau)=\frac{1}{\tau+1}\sum_{t=t_0}^{t_0+\tau}\sigma_i(t)\sigma_j(t). \tag{4}$$

The full model dynamics is then defined as follows.

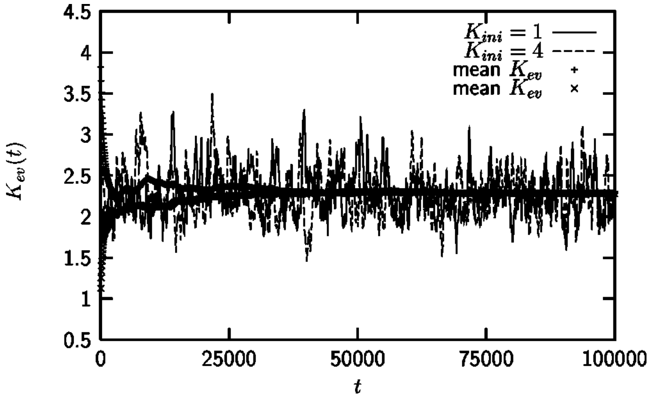

FIG. 1. Evolution of the average number of nonzero weights per neuron over evolutionary time, for a system of size $N=64$ (8×8) and two different initial connectivities ($K_{\mathrm{ini}}=1.0$ and $K_{\mathrm{ini}}=4.0$). Independent of the initial conditions, the networks evolve to a specific average connectivity. Parameters are $\beta=25$, $\epsilon=0.1$, a correlation cutoff $\alpha=0.8$, and an averaging time window of $\tau=200$.

(1) Start with a random network with an average connectivity (number of nonzero weights per neuron) K_{ini} and a random initial state vector $\vec{\sigma}(0)=(\sigma_1(0),\ldots,\sigma_N(0))$.

(2) For each neuron i, choose a random threshold θ_i from a Gaussian distribution of width ϵ and mean μ.

(3) Starting from the initial state, calculate the new system state applying Eq. (1) using parallel update. Iterate this for τ time steps.

(4) Randomly choose one neuron i and one of its neighbors j and determine the average correlation $C_{ij}(\tau/2)$ over the last $\tau/2$ time steps. (Alternatively, the correlation can be obtained from a synaptic variable providing a moving average at any given time).

(5) If $|C_{ij}(\tau)|$ is larger than a given threshold α, i receives a new link w_{ij} from site j with a weight chosen randomly from the interval $w_{ij}\in[-1,1]$.[3] If $|C_{ij}(\tau)|\leqslant\alpha$, the link w_{ij} is set to 0 (if nonzero.)

(6) Go to step 2 and iterate, using the current state of the network as new initial state.

The dynamics of this network is continuous in time, with neuron update on a fast time scale and topology update of the weights on a well-separated slow "synaptic plasticity" time scale. Note that the topology-changing rule does not involve any global knowledge, e.g., about attractors. A typical scenario of this dynamical evolution is shown in Fig. 1 where the average number of nonzero weights per neuron K_{ev} is shown as a time series and as cumulative mean.

One observes that the continuous network dynamics, including the slow local change of the topology, results in a convergence of the average connectivity of the network to a characteristic value which is independent of initial conditions.

Finite size scaling of the resulting average connectivity indicates the convergence towards a characteristic value for large network size N and exhibits the scaling relationship

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[2] The choice of the type of neighborhood is not critical, however, here the Moore neighborhood is more convenient than the von Neumann type since, in the latter case, the critical link density (fraction of nonzero weights) at the percolation threshold accidentally coincides with the attractor of the trivial developmental rule of producing a link with $p=0.5$. In general, also random sparse neighborhoods would work as demonstrated in Ref. [5].

[3] Also binary weights could be used as in Ref. [5].

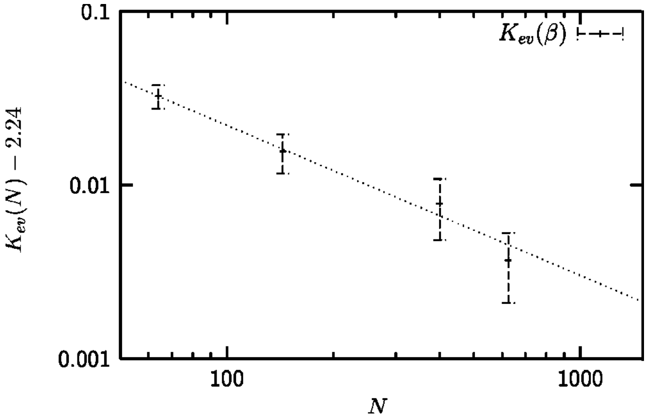

FIG. 2. Finite size scaling of the evolved average connectivity. Averages are taken over 4×10^5 time steps.

$$K_{\mathrm{ev}}(N)=aN^{-\delta}+b, \tag{5}$$

with $a=1.2\pm0.4$, $\delta=0.86\pm0.07$, and $b=2.24\pm0.03$. Thus, in the large system size limit $N\rightarrow\infty$ the networks evolve towards $K_{\mathrm{ev}}^{\infty}=2.24\pm0.03$ (see Fig. 2). The self-organization towards a specific average connectivity is largely insensitive to thermal noise of the network dynamics, up to $\approx$10% of thermal switching errors (or $\beta>10$) of the neurons. This indicates that the structure of a given dynamical attractor is robust against a large degree of noise. Figure 3 shows the evolved average connectivity as a function of the inverse temperature β.

While the stability of dynamical attractors on an intermediate time scale is an important requirement for the local sampling of neural correlation, on the long time scale of global topological changes, switching between attractors is necessary to ensure ergodicity at the attractor sampling level. The second source of noise, the slow random change in neural thresholds as defined in step (2) of the algorithm, is closely related to such transitions between attractors. While, in general, the model converges also when choosing some arbitrary fixed threshold θ and omitting step (2) from the algorithm, a small threshold noise facilitates transitions between limit cycle attractors [10] and thus improves sampling over all attractors of a network, resulting in an overall increased speed and robustness of the convergence. An asynchronous change of the threshold θ_i, updating one random θ_i after completing one sweep (time step) of the network, leads to similar results as the parallel rule defined above.

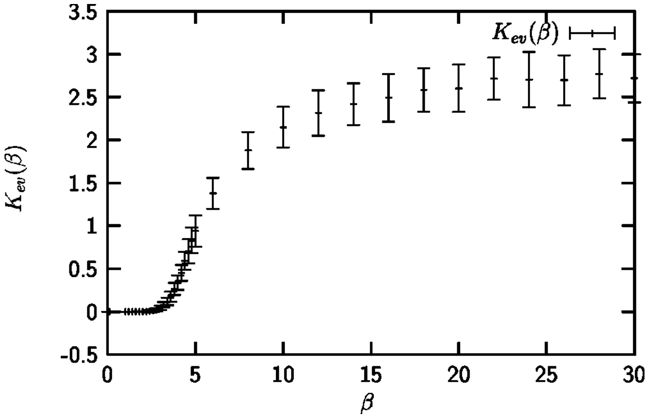

FIG. 3. Evolved average connectivity K_{ev} as a function of the inverse temperature β. Each point is averaged over 10^5 time steps in a network of size $N=64$ and $\alpha=0.5$.

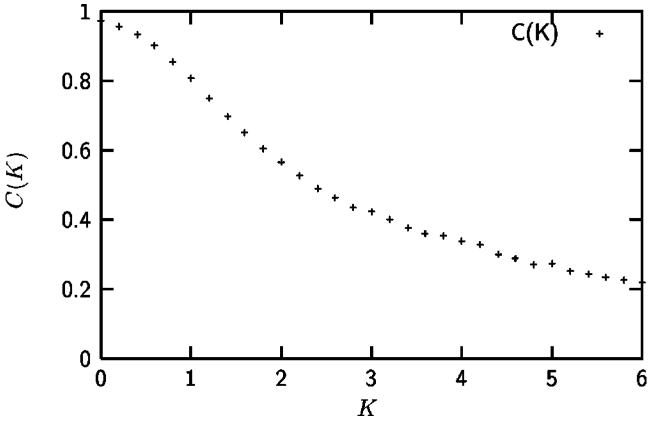

FIG. 4. The average correlation $|C_{ij}(\tau)|$ between random neurons of random networks at different connectivities K. Samples are taken over 1000 random networks with 100 random initial conditions each, for network size $N=64$.

The basic mechanism of the observed self-organization in this system is the weak coupling of topological change to an order parameter of the global dynamical state of the network, and thus is different from the mechanism of extremal dynamics, underlying many prominent models of self-organized criticality [11]. To illustrate this, let us for a moment consider the absolute average correlation $|C_{ij}(\tau)|$ of two neurons, which is the parameter used as a criterion for the rewiring process. For random networks, this quantity is shown in Fig. 4 for different conductivities K.

Note that the correlation is large for networks with small connectivity, and small for networks that are densely connected. The rewiring rule balances between these two regimes: For high correlation, it is more likely that a link is created, at low correlation, links are vanishing. The balance is reached most likely in the region of the curve where the slope reaches its maximum, as here the observed correlation reacts most sensitively to connectivity changes. As the steep portion of the correlation curve occurs in a region of small connectivities where also the critical connectivity $K_c\approx2$ of the network is located, this makes the correlation measure sensitive to the global dynamical state of the network and potentially useful as an approximation of the order parameter. Synaptic development dependent on averaged correlation between neurons can thus obtain approximate information about the global dynamical state of the network as is realized in the above toy model with a simple implementation on the basis of a threshold α. The exact choice of the threshold α is not critical, which can be seen from the histogram of the absolute correlation $|C_{ij}(\tau)|$ shown in Fig. 5 for a typical run of the model.

Correlations appear to cluster near high and near low values such that the cutoff can be placed anywhere inbetween the two regimes. Even a threshold value close to 1, as com-

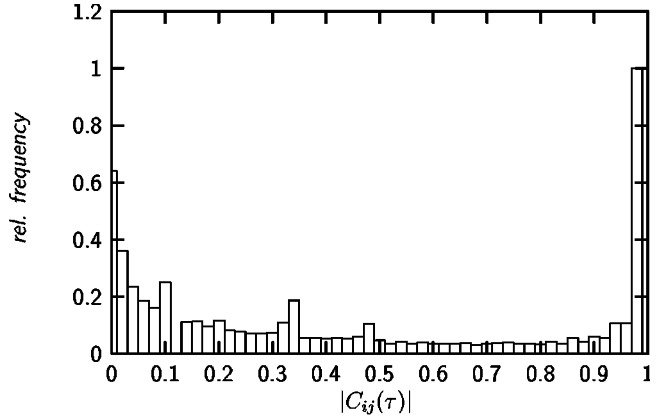

FIG. 5. Histogram of $|C_{ij}(\tau)|$ for a network evolving in time, with $N=64$ and $\beta=10$, taken over a run of 4×10^5 time steps.

pared with the correlation cutoff $\alpha=0.8$ used in the simulations here, only leads to a minor shift in K_{ev} and does not change the overall behavior.

Up to now we focused on changes of the network structure as a result of the dynamics on the network. A further aspect is how the structural changes affect the dynamics on the network itself. Do also dynamical observables of the networks self-organize as a result of the observed convergence of the network structure? An interesting quantity in this respect is the average length of periodic attractors as shown in Fig. 6.

Indeed, this dynamical observable of the network dynamics converges to a specific value independent of the initial network, similarly to the convergence of the structural parameter K considered earlier. From the K dependency of the neural pair correlation we have seen above that the rewiring criterion tends to favor connectivities near the critical connectivity of the network. Does also the evolved average attractor length relate to critical properties of the percolation transition? An approximate measure of this aspect is the finite size scaling of the evolved average period as shown in Fig. 7.

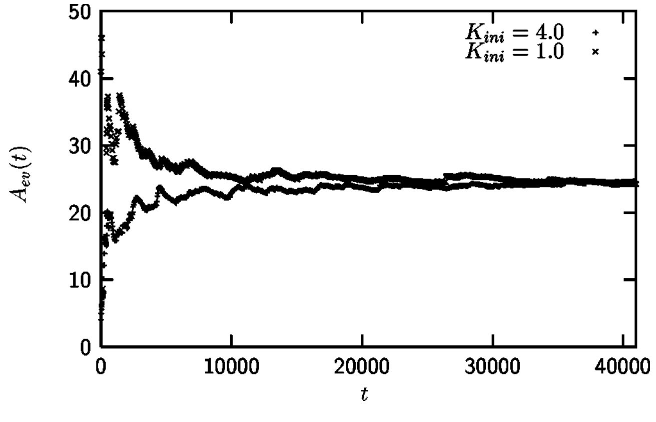

FIG. 6. Evolution of the cumulative average of attractor length for the same system as shown in Fig. 1. The mean attractor length converges to a value independent of the two initial conditions of the network shown here. The attractor length is measured at zero temperature in order to have an exactly defined measure.

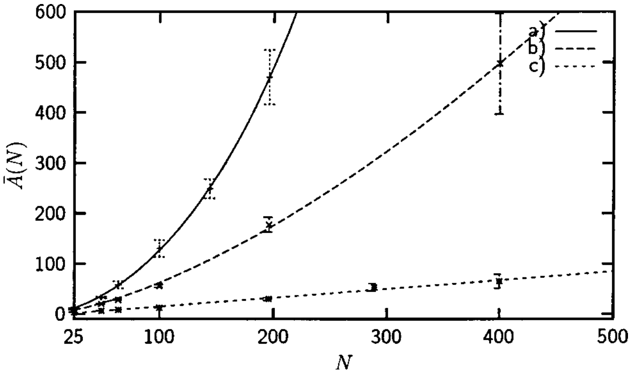

FIG. 7. Finite size scaling of the evolved average attractor period (b). Also shown for comparison is the corresponding scaling of the attractor lengths of an overcritical random network (a) with $K=3.8$ and an undercritical one (c) with $K=1.5$. Symbols denote measured values and lines correspond to the fits $f_a(x)=15.1x^{0.57}(e^{0.005x}-1)$, $f_b(x)=0.6x^{1.5}$, and $f_c(x)=0.28x^{0.75}\ln(0.097x)$.

For static networks we find that the attractor lengths typically scale exponentially with N in the overcritical regime, but less than linearly in the ordered regime. For the evolved connectivity K_{ev} in our model, we observe scaling close to criticality. Large evolved networks exhibit relatively short attractors, which otherwise for random networks in the overcritical regime could only be achieved by fine tuning. The self-organizing model studied here evolves nonchaotic networks without the need for parameter tuning.

To summarize, neural network development has been studied in an asymmetric model neural network. The developmental rule is based on local rewiring motivated by Hebbian, activity-dependent synaptic development. In a continuously running network, robust self-organization of the network towards the percolation transition between ordered and disordered dynamics is observed, independent of initial conditions and robust against thermal noise. The basic model is robust against changes in the details of the algorithm. We conclude that a weak coupling of the rewiring process to an approximate measurement of an order parameter of the global dynamics is sufficient for a robust self-organization towards criticality. In particular, the order parameter has been estimated solely from information available on the single synapse level via time averaging of correlated neural activities.

While here we considered self-organization in model neural networks, the observed mechanism may occur in other more complex systems. For example, global dynamical order from self-organization at the synapse level could, in principle, be at work in biological nervous systems as well. Prerequisites are an averaging procedure of correlated activities on slow time scales (similar to synaptic processes underlying learning through long term potentiation), and a coupling to synaptic development.

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